OPTIMIZING THE CAPACITY OF ORTHOGONAL AND BIORTHOGONAL DMT CHANNELS

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Abstract.[1] The uniform DFT filter bank has been used routinely in discrete multitone modulation (DMT) systems because of implementation efficiency. It has recently been shown that principal component filter banks (PCFB) which are known to be optimal for data compression and denoising applications, are also optimal for a number of criteria in DMT communication. In this paper we show that such filter banks are optimal even when scalar prefilters and postfilters are used around the channel. We show that the theoretically optimum scalar prefilter is the half-whitening solution, well known in data compression theory. We conclude with the observation that the PCFB continues to be optimal for the maximization of theoretical capacity as well.

1. INTRODUCTION

Figure 1 shows a maximally decimated analysis/synthesis system traditionally used in subband coding and signal denoising (all notations are as in [13]). In this paper we consider the dual of this configuration called the transmultiplexer, for application in discrete multitone modulation (DMT). Shown schematically in Fig. 2, this is an attractive method for data communication over twisted pair lines. It takes into account nonflat channels with possibly colored noise. The use of filter bank theory in the optimization of DMT systems has been of some interest in the past [8], [9]. We have shown recently [15] that the principal component filter bank or **PCFB** which is known to be optimal for many problems involving the subband coder [3] is also optimal for bit rate maximization in DMT systems. In this paper we extend this result in various directions. We show that the PCFB is the optimal orthonormal part even when scalar pre/post filters are inserted around the channel. The well-known **half whitening** filter is the optimum prefilter to be used in conjunction with the PCFB. We also show that the PCFB maximizes the theoretical capacity in DMT systems.

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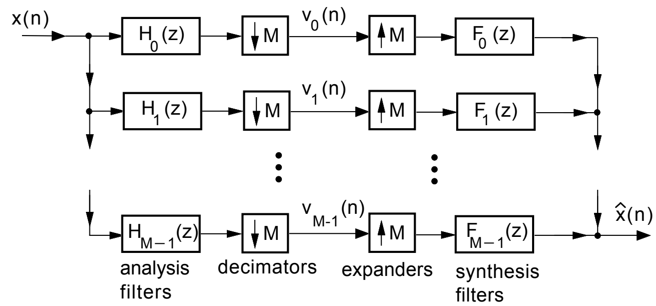

Fig. 1. The subband coder system.

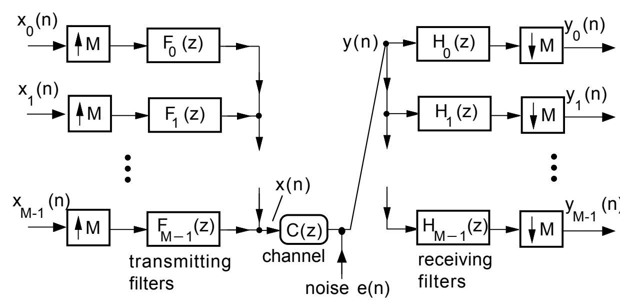

Fig. 2. The discrete multitone communication system.

2. PCFB REVIEW

The filter bank is **biorthogonal** if the filters are such that $H_k(e^{j\omega})F_m(e^{j\omega})\Big|_{\downarrow M} = \delta(k-m)$. This is equivalent to the perfect reconstruction property in Fig. 1, that is, $\widehat{x}(n) = x(n)$ for all n. The set of M filters $\{F_k(z)\}$ is said to be **orthonormal** or paraunitary if $F_k(e^{j\omega})F_m^*(e^{j\omega})|_{\downarrow M} = \delta(k-m)$. In this case biorthogonality or perfect reconstruction is achieved by the choosing $H_k(e^{j\omega}) = F_k^*(e^{j\omega})$. Consider two sets of M nonnegative numbers $\{a_n\}$ and $\{b_n\}$. We say that $\{a_n\}$ **majorizes** $\{b_n\}$ if, after reordering such that $a_n \geq a_{n+1}$ and $b_n \geq b_{n+1}$, we have

$$\sum_{n=0}^{P} a_n \geq \sum_{n=0}^{P} b_n$$

for $0 \leq P \leq M-1$, with equality for $P = M-1$. Thus all the partial sums in $\{a_n\}$ dominate those in $\{b_n\}$. Consider a given class $\mathcal{C}$ of M-band uniform orthonormal filter banks. This class can be the class $\mathcal{C}_{tc}$ of transform coders (filter lengths $\leq M$), or the class $\mathcal{C}_{ideal}$ of ideal filter banks (filters allowed to have infinite order, like brickwall filters). Or it could be a practically attractive class like the FIR class $\mathcal{C}_{fir}$ with filter orders bounded by a fixed integer. Given such a class $\mathcal{C}$ and an input power spectrum $S_{xx}(e^{j\omega})$ we say that a filter bank $\mathcal{F}$ in $\mathcal{C}$ is a principal component filter bank or **PCFB** if the set $\{p_k^2\}$ of its subband variances (i.e., variances of $v_k(n)$ in Fig. 1) majorizes the set $\{q_k^2\}$ of subband variances of all other filter banks in the class. The optimality of principal component filter banks (PCFBs) for various applications has been pointed out by a number of authors. A general result has been proved in [3] that any **concave** function ϕ of the subband variance vector

$$\mathbf{v} = [\,\sigma_{v_0}^2 \quad \sigma_{v_1}^2 \quad \ldots \quad \sigma_{v_{M-1}}^2\,]^T$$

is **minimized by a PCFB**. For the transform coder class $\mathcal{C}_{tc}$ and the ideal filter bank class $\mathcal{C}_{ideal}$, there is a systematic method to construct a PCFB. For other arbitrary classes it is possible that PCFBs do not exist [2]. Whenever we say that the PCFB is optimal for a problem, the implicit assumption is that the class of filter banks searched is such that a PCFB exists.

3. THE DMT COMMUNICATION SYSTEM

Figure 2 shows a schematic of discrete multitone communication (for background material see [4,5,7,12]). The signals $x_k(n)$ are b_k-bit symbols obtained from a PAM or QAM constellation [10]. Together these signals represent $\sum_k b_k = b$ bits, and are obtained from a b-bit block of a binary data stream. The symbols $x_k(n)$ are then interpolated M-fold by the filters $F_k(z)$. These filters cover different uniform regions of the frequency $0 \leq \omega \leq 2\pi$. The outputs of $F_k(z)$ can be regarded as modulated versions of the symbols. These are packed into M adjacent frequency bands (passbands of the filters) and added to obtain the composite signal $x(n)$. This is then sent through the channel which is represented in discrete time by a transfer function $C(z)$ and additive Gaussian noise $e(n)$ with power spectrum $S_{ee}(e^{j\omega})$. The received signal $y(n)$ is a distorted and noisy version of $x(n)$. The receiving filter bank $\{H_k(z)\}$ separates this signal into the components $y_k(n)$ which are distorted and noisy versions of the symbols $x_k(n)$. The detector estimates $x_k(n)$ from $y_k(n)$ with a certain error probability. If the filter bank $\{F_k, H_m\}$ is **biorthogonal** then we have the perfect reconstruction property $y_k(n) = x_k(n)$ in absence of channel imperfections (i.e., assuming $C(z) = 1$ and $e(n) = 0$). In practice we cannot assume this. We will assume that $\{F_k, H_m\}$ is biorthogonal and that the receiving filters are $H_k(z)/C(z)$ instead of $H_k(z)$, so that $C(z)$ is compensated for, or **equalized**, completely.

4. OPTIMAL DMT SYSTEMS AND PCFB

Assume that $x_k(n)$ are PAM symbols with 2^{b_k} equiprobable levels. The variance of $x_k(n)$ represents its **average power** P_k. The Gaussian channel noise $e(n)$ is filtered through $H_k(z)/C(z)$ and decimated by M. For the purpose of variance calculation, the model for the noise $q_k(n)$ at the detector input can therefore be taken as in Fig. 3. Let $\sigma_{q_k}^2$ be the variance of $q_k(n)$. Then the **probability of error** in detecting $x_k(n)$ is [10]

$$\mathcal{P}_e(k) = 2(1 - 2^{-b_k})\mathcal{Q}\left(\sqrt{\frac{3P_k}{(2^{2b_k}-1)\sigma_{q_k}^2}}\right) \qquad (1)$$

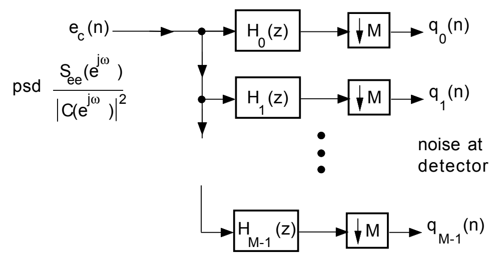

Fig. 3. A model for noise at detector input.

where $\mathcal{Q}(v) \stackrel{\Delta}{=} \int_v^\infty e^{-u^2/2} du/\sqrt{2\pi}$ (area of the normalized Gaussian tail). We can invert (1) to obtain P_k. The total transmitted power is then $P = \sum_{k=0}^{M-1} P_k = \sum_{k=0}^{M-1} \beta_k \times \sigma_{q_k}^2$ where β_k is some function of b_k and $\mathcal{P}_e(k)$. For fixed $\{b_k\}$ and $\{\mathcal{P}_e(k)\}$, the power P is a **concave** function of the noise variance vector

$$[\,\sigma_{q_0}^2 \quad \sigma_{q_1}^2 \quad \ldots \quad \sigma_{q_{M-1}}^2\,]^T \qquad (2)$$

Fig. 3 shows that this is the vector of subband variances for the orthonormal filter bank $\{H_k(e^{j\omega})\}$ in response to the power spectrum $S_{ee}(e^{j\omega})/|C(e^{j\omega})|^2$. The orthonormal filter bank $\{H_k(e^{j\omega})\}$ which **minimizes** $P = \sum_{k=0}^{M-1} \beta_k \times \sigma_{q_k}^2$ for fixed error probabilities and bit rates is therefore a **PCFB** for the power spectrum

$$S_{ee}(e^{j\omega})/|C(e^{j\omega})|^2$$

If we approximate $(1 - 2^{-b_k})$ with unity in Eq. (1) we can find an expression for b_k and show that the total bit rate $b = \sum b_k$ is

$$b = 0.5 \sum_{k=0}^{M-1} \log_2\Big(1 + \frac{3}{[\mathcal{Q}^{-1}(\mathcal{P}_e(k)/2)]^2} \frac{P_k}{\sigma_{q_k}^2}\Big) \qquad (3)$$

This is **convex** in the variance vector (2). Thus the orthonormal filter bank $\{H_k(e^{j\omega})\}$ which **maximizes bit rate** for fixed error probabilities and powers is again a PCFB for the power spectrum $S_{ee}(e^{j\omega})/|C(e^{j\omega})|^2$.

5. SCALAR PREFILTERING BEFORE CHANNEL

Consider again Fig. 2 where $\{H_k\}$ is orthonormal with $F_k(e^{j\omega}) = H_k^*(e^{j\omega})$. Assume as before than $C(z)$ has been equalized by inserting $1/C(z)$ (not shown). Suppose this configuration is now modified by insertion of a prefilter and postfilter around the channel (Fig. 4). Thus the effective transmitting filters are $F_k'(z) = F_k(z)D(z)$ and receiving filters are $H_k'(z) = H_k(z)/D(z)$. This defines a biorthogonal filter bank $\{F_k'(z), H_k'(z)\}$. This system can achieve better performance than the orthonormal system $\{F_k(z), H_k(z)\}$. For example we can shape $D(z)$ and $\{F_k(z)\}$ such that the transmitted power is minimized for fixed bit rates and probabilities of error.

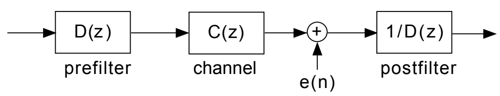

Fig. 4. Pre and post filters in the DMT system.

We assume that $x_k(n)$ has variance P_k for all n. The interpolated signal $s_k(n)$ (Fig. 5(a)) has a variance which in general depends on n. In fact if we assume that $x_k(n)$ is a WSS process, the signal $s_k(n)$ is cyclo WSS, and its variance is a periodic function of n with period M. The power in the kth symbol is this variance averaged over a period. To find this, redraw Fig. 5(a) as in Fig. 5(b) where $R_{nk}(z)$ are the polyphase components of $F_k(z)D(z)$. We shall assume that the symbols $x_k(n)$ are **white** with zero mean and variance P_k. This is consistent with the fact that $x_k(n)$ is generated by parsing a binary iid sequence [4]. Thus the variance at the output of $R_{nk}(z)$ is given by $\int P_k|R_{nk}(e^{j\omega})|^2 d\omega/2\pi$. The average variance of $s_k(n)$ is then

$$\begin{aligned} & \frac{P_k}{M}\sum_{n=0}^{M-1}\int_0^{2\pi}|R_{nk}(e^{j\omega})|^2 d\omega/2\pi \\ = & \frac{P_k}{M}\int_0^{2\pi}|F_k(e^{j\omega})D(e^{j\omega})|^2 d\omega/2\pi \end{aligned}$$

Assuming further that $x_k(n)$ are **uncorrelated** for different k, the total power input to the channel is the sum of these average variances:

$$P = \frac{1}{M}\sum_{k=0}^{M-1} P_k \int_0^{2\pi}|F_k(e^{j\omega})D(e^{j\omega})|^2 d\omega/2\pi \qquad (4)$$

The quantity P_k is also the physical signal-power at the input of the detector. The noise variance $\sigma_{q_k}^2$ at the detector input can be computed by referrring to Fig. 3 and inserting the additional factor $1/D(z)$ in the noise transfer functions. Thus

$$\sigma_{q_k}^2 = \int_0^{2\pi}\frac{S_{ee}(e^{j\omega})|H_k(e^{j\omega})|^2}{|C(e^{j\omega})D(e^{j\omega})|^2} d\omega/2\pi$$

Since $P_k = g[b_k, \mathcal{P}_e(k)]\sigma_{q_k}^2$ for some $g[.,.]$, the total power is

$$\begin{aligned} P &= \frac{1}{M}\sum_{k=0}^{M-1} g[b_k, \mathcal{P}_e(k)]\int_0^{2\pi}\frac{S_{ee}(e^{j\omega})|F_k(e^{j\omega})|^2}{|C(e^{j\omega})D(e^{j\omega})|^2}\frac{d\omega}{2\pi} \\ &\times \int_0^{2\pi}|F_k(e^{j\omega})D(e^{j\omega})|^2\frac{d\omega}{2\pi} \end{aligned}$$

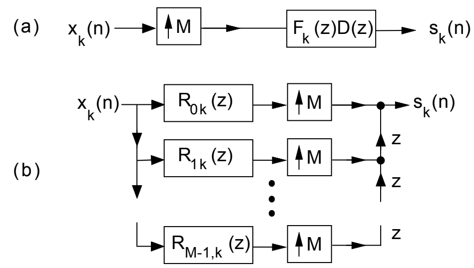

Fig. 5. (a) The kth band signal $s_k(n)$ actually going into the channel, and (b) the polyphase representation.

where we have substituted the preceding expression for $\sigma_{q_k}^2$ and used the fact that $H_k(e^{j\omega}) = F_k^*(e^{j\omega})$ for any orthonormal PRFB. For a given channel, $C(e^{j\omega})$ and $S_{ee}(e^{j\omega})$ are fixed. Assume the set of error probablities $\{\mathcal{P}_e(k)\}$ and bit rates $\{b_k\}$ are also fixed. The total power input to the channel then depends on the orthonormal filter bank $\{F_k(e^{j\omega})\}$ and the prefilter $D(e^{j\omega})$. The next result shows how this power can be minimized.

Theorem 1. *Optimum prefiltered orthonormal FB for DMT*. Assume that the modulation symbols $x_k(n)$ are white, and uncorrelated for different k. For fixed probabilities of error $\mathcal{P}_e(k)$ and bit rates b_k, the combination of orthonormal filter bank $\{F_k\}$ and prefilter $D(z)$ which minimizes the total required power P is obtained as follows: (a) Choose $D(z)$ with magnitude response (5) and (b) make $\{F_k\}$ = PCFB for $\sqrt{S_{ee}(e^{j\omega})}/|C(e^{j\omega})|$. $\diamondsuit$

Proof. From Cauchy-Schwartz inequality we have

$$\int\frac{S_{ee}|F_k|^2}{|CD|^2}\frac{d\omega}{2\pi}\int|F_kD|^2\frac{d\omega}{2\pi} \geq \Big(\int\frac{\sqrt{S_{ee}}|F_k|^2}{|C|}\frac{d\omega}{2\pi}\Big)^2$$

where the argument $(e^{j\omega})$ has been eliminated for simplicity. Equality holds when the two integrands on the left are equal, that is,

$$|D(e^{j\omega})| = \Big(S_{ee}(e^{j\omega})/|C(e^{j\omega})|^2\Big)^{1/4} \qquad (5)$$

This is the optimum $|D(e^{j\omega})|$ no matter what the orthonormal filter bank $\{F_k\}$ is. With the prefilter chosen as above, the total transmitted power is $P = \sum_{k=0}^{M-1} g[b_k, \mathcal{P}_e(k)]\eta_k^2/M$ where

$$\eta_k^2 = \left(\int_0^{2\pi} \frac{\sqrt{S_{ee}(e^{j\omega})}|F_k(e^{j\omega})|^2}{|C(e^{j\omega})|} d\omega/2\pi \right)^2$$

Thus P is a concave function of $[\,\eta_0^2 \quad \eta_1^2 \quad \ldots \quad \eta_{M-1}^2\,]^T$ which can be regarded as a subband variance vector from an orthonormal analysis bank with input power spectrum $\sqrt{S_{ee}(e^{j\omega})}/|C(e^{j\omega})|$. Applying the result of Sec. 2 we conclude that the orthonormal filter bank $\{F_k\}$ minimizing the total power is a PCFB for the power spectrum $\sqrt{S_{ee}(e^{j\omega})}/|C(e^{j\omega})|$. $\bigtriangledown \bigtriangledown \bigtriangledown$

The solution (5) also arises in optimal prefiltering prior to scalar quantization, and is said to be the **half whitening** filter [13] for the spectrum $|C(e^{j\omega})|^2/S_{ee}(e^{j\omega})$.

6. CONCLUDING REMARKS

We conclude by observing some similarities and differences between the actual bit rate (3) and the theoretical capacity of the DMT system. The biorthogonal DMT system with ideal channel equalizer can be represented by the model shown in Fig. 6 where $x_k(n)$ are the modulation symbols and $q_k(n)$ the noise components shown in Fig. 3. In general it is not true that the effective noise components $q_k(n)$ are Gaussian, white, and uncorrelated. However if the number of bands M is large and the filters $H_k(z)$ are good approximations to ideal filters then this is nearly the case. In this case the channel shown in Fig. 6 is identical to the parallel Gaussian channel and has capacity [6]

$$\mathcal{C} = 0.5 \sum_{k=0}^{M-1} \log_2\Big(1 + \frac{P_k}{\sigma_{q_k}^2}\Big) \tag{6}$$

Since the noise variances $\sigma_{q_k}^2$ depend on the filters $\{F_k, H_k\}$, the above capacity $\mathcal{C}$ also depends on them. For the case where $\{F_k\}$ is an orthonormal filter bank this **capacity is maximized** if $\{F_k\}$ is chosen as a PCFB for the power spectrum $S_{ee}(e^{j\omega})/|C(e^{j\omega})|^2$. The reason again is that (6) is convex in the variance vector (2). Moreover, as in [6], we can optimally allocate the powers P_k under a power constraint $P = \sum_k P_k$.

Equation (3) is the **bit rate achieved** for fixed probabilities of error $\{\mathcal{P}_e(k)\}$, and without channel-coding in subbands. Eq. (6) is the **information capacity**, that is, the theoretical upper bound on achievable bit rate with arbitrarily small error. We see that both (3) and (6) depend on the choice of filter bank, and are maximized by the PCFB. Suppose the error probabilities are $\mathcal{P}_e(k) = 10^{-7}$ for all k. A calculation of the factor $3/[\mathcal{Q}^{-1}(\mathcal{P}_e(k)/2)]^2$ shows that if the two quantities b and $\mathcal{C}$ have to be equal then the total power in (3) should be **9.74 dB** more than the power used in (6). Channel coding is included in many DMT systems in order to reduce this gap.

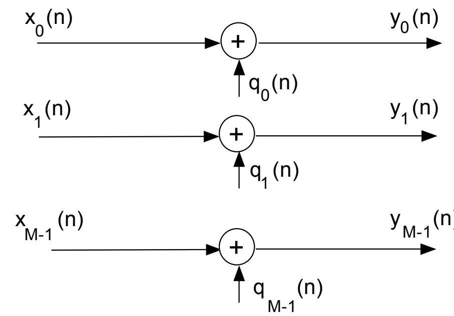

Fig. 6. Equivalent DMT system for noise analysis.

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